

Almost Periodic Dynamics of Perturbed Infinite-Dimensional Dynamical Systems

Bixiang Wang *

Department of Mathematics, New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA
Email: bwang@nmt.edu

Abstract. This paper is concerned with the dynamics of an infinite-dimensional gradient system under small almost periodic perturbations. Under the assumption that the original autonomous system has a global attractor given as the union of unstable manifolds of a finite number of hyperbolic equilibrium solutions, we prove that the perturbed non-autonomous system has exactly the same number of almost periodic solutions. As a consequence, the pullback attractor of the perturbed system is given by the union of unstable manifolds of these finitely many almost periodic solutions. An application of the result to the Chafee-Infante equation is discussed.

Key words. Almost periodic solution, hyperbolic solution, pullback attractor.

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1 Introduction

In this paper, we study the dynamics of an infinite-dimensional gradient system under almost periodic perturbations. Let A_0 be a sectorial operator in a Banach space X and X^α be the fractional powers of X with $0 \leq \alpha < 1$. Consider the autonomous nonlinear equation

$$\frac{dx}{dt} + A_0 x = f(x), \quad x(0) = x_0, \quad (1.1)$$

where $f : X^\alpha \rightarrow X$ is locally Lipschitz continuous. Suppose for each $x_0 \in X^\alpha$, the initial-value problem (1.1) has a unique solution $x \in C([0, \infty); X^\alpha)$. Assume further that equation (1.1) has a Liapunov function and only a finite number of equilibrium solutions x_i^* , $1 \leq i \leq n$. If problem

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(1.1) has a global attractor \mathcal{A} , then it is well known (see, e.g., [1]) that \mathcal{A} is given by the union of unstable manifolds of all the equilibrium solutions, i.e.,

$$\mathcal{A} = \bigcup_{i=1}^n W^u(x_i^*), \quad (1.2)$$

where $W^u(x_i^*)$ is the unstable manifold of x_i^* .

In this paper, we want to explore the effect of almost periodic perturbations on the structure of the attractor \mathcal{A} . More precisely, given $\epsilon > 0$, consider the non-autonomously perturbed equation

$$\frac{dx}{dt} + A_0 x = f(x) + g_\epsilon(t, x), \quad x(\tau) = x_0, \quad (1.3)$$

where $g_\epsilon(t, x) : \mathbb{R} \times X^\alpha \rightarrow X$ is an almost periodic function in t uniformly in x , and $g_\epsilon \rightarrow 0$ in a sense (which will be made clear in the next section). Under certain conditions, we will prove that, for every small $\epsilon > 0$, problem (1.3) has exactly n almost periodic solutions $\phi_{i,\epsilon}^*$, $i \leq i \leq n$, and each $\phi_{i,\epsilon}^*$ corresponds to an equilibrium solution x_i^* of problem (1.1). This result along with [2] implies that the pullback attractor $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$ of problem (1.3) can be characterized by

$$\mathcal{A}_\epsilon(t) = \bigcup_{i=1}^n W_\epsilon^u(\phi_{i,\epsilon}^*)(t), \quad t \in \mathbb{R}, \quad (1.4)$$

where $W_\epsilon^u(\phi_{i,\epsilon}^*)$ is the unstable manifold of the almost periodic solution $\phi_{i,\epsilon}^*$, which will be defined in Section 2. By (1.2) and (1.4) we see that the structure of the attractor \mathcal{A} is preserved under a small almost periodic perturbation, and the almost periodic solutions $\phi_{i,\epsilon}^*$ play exactly the same role on the dynamics of the perturbed system as the equilibrium solutions x_i^* do on the autonomous equation. In this sense, almost periodic solutions are appropriate extension of equilibrium solutions to almost periodic systems. In particular, when the perturbation g_ϵ is time-periodic, it can be proved that the almost periodic solutions $\phi_{i,\epsilon}^*$ are actually periodic solutions. In this case, the dynamics of the corresponding periodic systems is completely governed by a set of finitely many periodic solutions.

Closely related to the present paper, a more general non-autonomous perturbation problem was recently studied by Carvalho et. al. in [2, 3], where the perturbation g_ϵ was not assumed to be almost periodic in time. In that case, the authors proved that, for each i with $1 \leq i \leq n$ and each small $\epsilon > 0$, the equation (1.3) has a complete solution $\xi_{i,\epsilon}^*$ (i.e., $\xi_{i,\epsilon}^*$ is defined for all $t \in \mathbb{R}$) which corresponds to the equilibrium solution x_i^* of equation (1.1). Further, the authors proved that the pullback attractor $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$ of the perturbed equation is given by

$$\mathcal{A}_\epsilon(t) = \bigcup_{i=1}^n W_\epsilon^u(\xi_{i,\epsilon}^*)(t), \quad t \in \mathbb{R}, \quad (1.5)$$

where $W_\epsilon^u(\xi_{i,\epsilon}^*)$ is the unstable manifold of $\xi_{i,\epsilon}^*$. This is a remarkable result on the exact structure of pullback attractors of an infinite-dimensional non-autonomous system. Note that, in the nontrivial case, the non-autonomous equation (1.3) has infinitely many complete solutions. In fact, every solution starting from the attractor is a complete solution. The authors of [2] successfully identified a finite number of complete solutions which are crucial in determining the structure of the pullback attractor. In this paper, we will prove that the complete solution $\xi_{i,\epsilon}^*$ ($1 \leq i \leq n$) founded in [2] are actually almost periodic solutions when the perturbation g_ϵ is uniformly almost periodic. We further demonstrate that problem (1.3) has no other almost periodic solutions except those $\xi_{i,\epsilon}^*$. The convergence of solutions of the perturbed equation was also established in [2]. Particularly, they proved that every solution of the equation converges to one of the complete bounded solutions $\xi_{i,\epsilon}^*$ as $t \rightarrow \infty$.

The solutions and dynamics of almost periodic differential equations have long been investigated in the literature, see, e.g., [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Particularly, the ω -limit sets of almost periodic equations have been studied by the authors of [7, 8, 9, 10, 11].

In the next section, we will recall some concepts on sectorial operators and uniformly almost periodic functions. We will present our main results in this section. Section 3 is devoted to the proof of the results. We first show the existence of almost periodic solutions for the non-autonomously perturbed equation, and then prove the number of almost periodic solutions is finite. In the last section, we discuss an application of our results to the Chafee-Infante equation under almost periodic perturbations.

2 Notation and Main Results

Let A_0 be a sectorial operator in a Banach space X with norm $\|\cdot\|$. Suppose there is a number $a \in \mathbb{R}$ such that the fractional powers $(A_0 + aI)^\alpha$ are well defined for all $\alpha \in \mathbb{R}$. Set $A_1 = A_0 + aI$ and $X^\alpha = D(A_1^\alpha)$, the domain of A_1^α , for $\alpha \geq 0$. Then it is known (see, e.g., [16]) that X^α is a Banach space with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ for $x \in D(A_1^\alpha)$. From now on, we assume $\alpha \in [0, 1)$ and $f : X^\alpha \rightarrow X$ is a continuously differentiable function that is Lipschitz continuous in bounded subsets of X^α . Consider the nonlinear autonomous equation

$$\frac{dx}{dt} + A_0 x = f(x), \quad t > 0 \quad \text{and} \quad x(0) = x_0 \in X^\alpha. \quad (2.1)$$

Suppose Problem (2.1) is well-posed in X^α , that is, for each $x_0 \in X^\alpha$ the initial-value problem has a unique solution $x \in C([0, \infty); X^\alpha)$ which depends continuously on initial data x_0 in X^α .

Let $\{S(t)\}_{t \geq 0}$ be the evolution semigroup generated by problem (2.1), that is, for every $t \geq 0$ and $x_0 \in X^\alpha$, $S(t)x_0 = x(t, x_0)$, the solution of equation (2.1) at time t with initial condition x_0 . If $x_0^* \in X^\alpha$ and $S(t)x_0^* = x_0^*$ for all $t \geq 0$, then x_0^* is called an equilibrium solution of $\{S(t)\}_{t \geq 0}$. An equilibrium solution x_0^* is said to be hyperbolic if the spectrum of $A = A_0 - f'(x_0^*)$ does not intersect the imaginary axis, and there is a projection $P : X \rightarrow X$ such that the following conditions are satisfied, for some positive numbers M and β :

$$e^{-At}P = Pe^{-At}, \quad \forall t \geq 0. \quad (2.2)$$

$$\|e^{-At}Px\| \leq Me^{\beta t}\|x\|, \quad \forall x \in X, \forall t \leq 0. \quad (2.3)$$

$$\|e^{-At}(I - P)x\| \leq Me^{-\beta t}\|x\|, \quad \forall x \in X, \forall t \geq 0. \quad (2.4)$$

Then it follows from Lemma 7.6.2 in [16] that there is a positive number M_1 such that for all $x \in X$:

$$\|e^{-A(t-s)}Px\|_\alpha \leq M_1 e^{\beta(t-s)}\|x\|, \quad \forall t < s, \quad (2.5)$$

$$\|e^{-A(t-s)}(I - P)x\|_\alpha \leq M_1 e^{-\beta(t-s)} \max\{1, (t-s)^{-\alpha}\}\|x\|, \quad \forall t > s. \quad (2.6)$$

Throughout this paper, we assume that $\{S(t)\}_{t \in \mathbb{R}}$ has the following properties:

- (H1) $\{S(t)\}_{t \in \mathbb{R}}$ has a global attractor \mathcal{A} in X^α .
- (H2) $\{S(t)\}_{t \in \mathbb{R}}$ has only a finite number of hyperbolic equilibrium solutions x_i^* , $1 \leq i \leq n$.
- (H3) $\{S(t)\}_{t \in \mathbb{R}}$ has a Liapunov function in X^α .

Under these conditions it is well known (see, e.g., [1]) that the attractor \mathcal{A} has the structure

$$\mathcal{A} = \bigcup_{i=1}^n W^u(x_i^*), \quad (2.7)$$

where $W^u(x_i^*)$ is the unstable manifold of x_i^* . We intend to examine the behavior of this attractor under small almost periodic perturbations. Recall that a continuous function $g : \mathbb{R} \rightarrow X$ is called an almost periodic function if for every $\delta > 0$ there exists a positive number l (depending on δ) such that every interval I of length l contains a number s for which $\|g(t+s) - g(t)\| < \delta$ for all $t \in \mathbb{R}$. Every almost periodic function g is bounded, i.e., $g \in C_b(\mathbb{R}, X)$, where $C_b(\mathbb{R}, X)$ is the Banach space of all continuous and bounded functions from \mathbb{R} to X with norm $\|g\|_{C_b(\mathbb{R}, X)} = \sup_{t \in \mathbb{R}} \|g(t)\|$. A continuous function $g(t, x) : \mathbb{R} \times X^\alpha \rightarrow X$ is said to be almost periodic in t uniformly in x if for every $\delta > 0$ and every compact subset K of X^α there exists a positive number l (depending on δ and K) such that every interval I of length l contains a number s for which

$$\|g(t+s, x) - g(t, x)\| < \delta \quad \text{for all } t \in \mathbb{R} \quad \text{and } x \in K.$$

Given $\epsilon \in [0, 1)$, let $g_\epsilon(t, x) : \mathbb{R} \times X^\alpha \rightarrow X$ be almost periodic in t uniformly in x . We further assume that g_ϵ is continuously differentiable in $x \in X^\alpha$ and satisfies, for every $r > 0$,

$$(H4) \quad \limsup_{\epsilon \rightarrow 0} \sup_{t \in \mathbb{R}} \sup_{\|x\|_\alpha \leq r} \left(\|g_\epsilon(t, x)\| + \left\| \frac{\partial g_\epsilon}{\partial x}(t, x) \right\|_{L(X^\alpha, X)} \right) = 0.$$

Consider the equation with almost periodic perturbations

$$\frac{dx}{dt} + A_0 x = f(x) + g_\epsilon(t, x), \quad x(\tau) = x_0 \in X^\alpha, \quad (2.8)$$

where $t > \tau$ with $\tau \in \mathbb{R}$. Suppose, for each $x_0 \in X^\alpha$, the initial-value problem (2.8) has a unique solution $x \in C([\tau, \infty); X^\alpha)$ which depends continuously on initial data x_0 in X^α . Let $\{S_\epsilon(t, \tau) : \tau \in \mathbb{R}, t \geq \tau\}$ be the evolution process generated by the non-autonomous equation (2.8), that is, for every $\tau \in \mathbb{R}$, $t \geq \tau$ and $x_0 \in X^\alpha$, $S_\epsilon(t, \tau)x_0 = x_\epsilon(t, \tau, x_0)$, the solution of equation (2.8) with $x_\epsilon(\tau, \tau, x_0) = x_0$. For convenience, we also write the process as $S_\epsilon(\cdot, \cdot)$ occasionally.

A family of compact subsets of X^α , $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$, is called a pullback attractor of $S_\epsilon(\cdot, \cdot)$ if the following conditions are fulfilled:

- (1) $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$ is invariant, i.e., for all $\tau \in \mathbb{R}$ and $t \geq \tau$, $S_\epsilon(t, \tau)\mathcal{A}_\epsilon(\tau) = \mathcal{A}_\epsilon(t)$.
- (2) $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$ attracts all bounded subsets B of X^α , i.e., $\text{dist}(S_\epsilon(t, \tau)B, \mathcal{A}_\epsilon(t)) \rightarrow 0$ as $\tau \rightarrow -\infty$, where the Hausdorff semi-distance in X^α is used.

Suppose $S_\epsilon(\cdot, \cdot)$ has a pullback attractor $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$. We will characterize the structure of this attractor in the present paper. To the end, we further assume that there are $\epsilon_0 > 0$ and a compact subset K of X^α such that

$$(H5) \quad \bigcup_{\epsilon \leq \epsilon_0} \bigcup_{t \in \mathbb{R}} \mathcal{A}_\epsilon(t) \subseteq K.$$

Under assumptions (H1)-(H5) we will prove that problem (2.8) has exactly n almost periodic solutions and the pullback attractor is the union of unstable manifolds of all the almost periodic solutions. Our main results are summarized as follows.

Theorem 2.1. *Suppose (H1)-(H5) hold and $g_\epsilon(t, x)$ is almost periodic in $t \in \mathbb{R}$ uniformly in $x \in X^\alpha$. Then there is $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ problem (2.8) has exactly n almost periodic solutions $\phi_{i,\epsilon}^*$, $1 \leq i \leq n$. Further, each $\phi_{i,\epsilon}^*$ corresponds to the equilibrium solution x_i^* of problem (2.1) in the sense:*

$$\limsup_{\epsilon \rightarrow 0} \sup_{t \in \mathbb{R}} \|\phi_{i,\epsilon}^*(t) - x_i^*\|_\alpha = 0, \quad \forall 1 \leq i \leq n.$$

As a consequence of this result, it follows from Theorem 2.11 of [2] that the pullback attractor

of problem (2.8) can be characterized by the union of unstable manifolds of the almost periodic solutions. More precisely, we have:

Corollary 2.2. *Suppose (H1)-(H5) hold and $g_\epsilon(t, x)$ is almost periodic in $t \in \mathbb{R}$ uniformly in $x \in X^\alpha$. Then there is $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$:*

(1) *The pullback attractor $\{\mathcal{A}_\epsilon(\tau)\}_{\tau \in \mathbb{R}}$ of problem (2.8) is given by*

$$\mathcal{A}_\epsilon(\tau) = \bigcup_{i=1}^n W_\epsilon^u(\phi_{i,\epsilon}^*)(\tau), \quad \tau \in \mathbb{R},$$

where $W_\epsilon^u(\phi_{i,\epsilon}^)(\tau)$ consists of all $x_0 \in X^\alpha$ such that there is a backwards solution $x(t, \tau, x_0)$ of problem (2.8) for which $x(\tau, \tau, x_0) = x_0$ and $\|x(t, \tau, x_0) - \phi_{i,\epsilon}^*(t)\|_\alpha \rightarrow 0$ as $t \rightarrow -\infty$. In addition, for every $\tau \in \mathbb{R}$, the Hausdorff dimension of $\mathcal{A}_\epsilon(\tau)$ is the same as that of \mathcal{A} .*

(2) *For every $\tau \in \mathbb{R}$ and $x_0 \in X^\alpha$ there exists $i \in \{1, \dots, n\}$ such that $\|S_\epsilon(t, \tau)x_0 - \phi_{i,\epsilon}^*(t)\|_\alpha \rightarrow 0$ as $t \rightarrow \infty$. In addition, if $S(t, \tau)x_0$ is a complete bounded solution, then there is $j \neq i$ such that $\|S_\epsilon(t, \tau)x_0 - \phi_{j,\epsilon}^*(t)\|_\alpha \rightarrow 0$ as $t \rightarrow -\infty$.*

3 Proof of Main Results

This section is devoted to the proof of our main results. We first prove the existence of almost periodic solutions of problem (2.8).

Lemma 3.1. *Suppose (H4) holds and $g_\epsilon(t, x)$ is almost periodic in $t \in \mathbb{R}$ uniformly in $x \in X^\alpha$. Then for every hyperbolic equilibrium solution x_0^* of problem (2.1), there are positive numbers δ_0 and ϵ_0 such that for each $\epsilon \in (0, \epsilon_0)$, problem (2.8) has a unique almost periodic solution $x_\epsilon^* : \mathbb{R} \rightarrow X^\alpha$ which satisfies $\|x_\epsilon^* - x_0^*\|_{C_b(\mathbb{R}, X^\alpha)} \leq \delta_0$. Furthermore, $x_\epsilon^* \rightarrow x_0^*$ in $C_b(\mathbb{R}, X^\alpha)$ as $\epsilon \rightarrow 0$.*

Proof. Suppose that $x : \mathbb{R} \rightarrow X^\alpha$ is an almost periodic solution of equation (2.8). Then for $y = x - x_0^*$, $\tau \in \mathbb{R}$ and $t \geq \tau$ we have

$$\frac{dy}{dt} + Ay = h(y) + g_\epsilon(t, x_0^* + y), \quad (3.1)$$

where $A = A_0 - f'(x_0^*)$ and $h(y) = f(y + x_0^*) - f(x_0^*) - f'(x_0^*)y$. Note that $h(0) = 0$ and $h'(0) = 0$. It follows from (3.1) that, for all $t \geq \tau$ with $\tau \in \mathbb{R}$,

$$y(t) = e^{-A(t-\tau)}y(\tau) + \int_\tau^t e^{-A(t-s)}(h(y(s)) + g_\epsilon(s, x_0^* + y(s))) ds. \quad (3.2)$$

Since x_0^* is a hyperbolic equilibrium solution of problem (2.1), there is a projection P satisfying (2.2)-(2.6). Applying P and $I - P$ to (3.2) we find that

$$Py(t) = e^{-A(t-\tau)}Py(\tau) + \int_{\tau}^t e^{-A(t-s)}P(h(y(s)) + g_{\epsilon}(s, x_0^* + y(s))) ds, \quad (3.3)$$

and

$$(I - P)y(t) = e^{-A(t-\tau)}(I - P)y(\tau) + \int_{\tau}^t e^{-A(t-s)}(I - P)(h(y(s)) + g_{\epsilon}(s, x_0^* + y(s))) ds. \quad (3.4)$$

Note that $y : \mathbb{R} \rightarrow X^{\alpha}$ is bounded since it is almost periodic. Letting $\tau \rightarrow +\infty$ and $\tau \rightarrow -\infty$ in (3.3) and (3.4), respectively, by (2.3) and (2.4) we obtain

$$Py(t) = - \int_t^{\infty} e^{-A(t-s)}P(h(y(s)) + g_{\epsilon}(s, x_0^* + y(s))) ds,$$

and

$$(I - P)y(t) = \int_{-\infty}^t e^{-A(t-s)}(I - P)(h(y(s)) + g_{\epsilon}(s, x_0^* + y(s))) ds.$$

Therefore, y must satisfy the equation, for all $t \in \mathbb{R}$,

$$y(t) = \int_{-\infty}^t e^{-A(t-s)}(I - P)\phi_{\epsilon}(s, y(s))ds - \int_t^{\infty} e^{-A(t-s)}P\phi_{\epsilon}(s, y(s))ds, \quad (3.5)$$

where $\phi_{\epsilon}(t, x) = h(x) + g_{\epsilon}(t, x_0^* + x)$ for $t \in \mathbb{R}$ and $x \in X^{\alpha}$. Conversely, if $y : \mathbb{R} \rightarrow X^{\alpha}$ is almost periodic and fulfills (3.5), then y is an almost periodic solution of equation (2.8). So finding an almost periodic solution of equation (2.8) amounts to finding a fixed point of the mapping \mathcal{F} given by

$$\mathcal{F}(y)(t) = \int_{-\infty}^t e^{-A(t-s)}(I - P)\phi_{\epsilon}(s, y(s))ds - \int_t^{\infty} e^{-A(t-s)}P\phi_{\epsilon}(s, y(s))ds. \quad (3.6)$$

Given $\delta > 0$, set

$$Z = \{y : \mathbb{R} \rightarrow X^{\alpha}, y \text{ is almost periodic and } \sup_{t \in \mathbb{R}} \|y(t)\|_{\alpha} \leq \delta\}.$$

Then Z is a complete metric space with distance induced by the norm of $C_b(\mathbb{R}, X^{\alpha})$. In what follows, we will prove, for a sufficiently small δ , \mathcal{F} has a unique fixed point in Z .

Note that h is continuously differentiable and $h'(0) = 0$. So there is $\delta_1 > 0$ such that for all $x \in X^{\alpha}$ with $\|x\|_{\alpha} < \delta_1$,

$$\|h'(x)\|_{L(X^{\alpha}, X)} < \min \left\{ \frac{\beta}{8M_1}, \frac{\beta^{1-\alpha}}{8M_1\Gamma(1-\alpha)}, \frac{1}{2M_1(4\beta^{-1} + 2\beta^{\alpha-1}\Gamma(1-\alpha))} \right\}, \quad (3.7)$$

where β and M_1 are the positive constants in (2.5) and (2.6), and $\Gamma(\alpha)$ is the value of the Γ function at α . Let $\delta_0 = \min\{1, \delta_1\}$. Then given $\delta \in (0, \delta_0]$, by (H4) we find that there is $\epsilon_0 > 0$ depending on δ such that for all $\epsilon < \epsilon_0$,

$$\sup_{t \in \mathbb{R}} \sup_{\|x\|_\alpha \leq 1 + \|x_0^*\|_\alpha} \|g_\epsilon(t, x)\| < \min \left\{ \frac{\beta\delta}{8M_1}, \frac{\beta^{1-\alpha}\delta}{8M_1\Gamma(1-\alpha)} \right\}, \quad (3.8)$$

and

$$\sup_{t \in \mathbb{R}} \sup_{\|x\|_\alpha \leq 1 + \|x_0^*\|_\alpha} \left\| \frac{\partial g_\epsilon}{\partial x}(t, x) \right\|_{L(X^\alpha, X)} < \frac{1}{2M_1(4\beta^{-1} + 2\beta^{\alpha-1}\Gamma(1-\alpha))}. \quad (3.9)$$

Given $y \in Z$ it follows from (2.5)-(2.6) and (3.6) that

$$\|\mathcal{F}(y)(t)\|_\alpha \leq M_1 \int_t^\infty e^{\beta(t-s)} \|\phi_\epsilon(s, y(s))\| ds + M_1 \int_{-\infty}^t e^{-\beta(t-s)} (1 + (t-s)^{-\alpha}) \|\phi_\epsilon(s, y(s))\| ds. \quad (3.10)$$

The right-hand side of (3.10) is estimated as follows. Since $y \in Z$ and $\delta < \delta_1$, by (3.7) we get, for all $s \in \mathbb{R}$,

$$\|h(y(s))\| = \|h(y(s)) - h(0)\| \leq \sup_{\|x\|_\alpha \leq \delta} \|h'(x)\|_{L(X^\alpha, X)} \sup_{s \in \mathbb{R}} \|y(s)\|_\alpha \leq \min \left\{ \frac{\beta\delta}{8M_1}, \frac{\beta^{1-\alpha}\delta}{8M_1\Gamma(1-\alpha)} \right\},$$

which along with (3.8) shows that, for all $\epsilon < \epsilon_0$,

$$\sup_{s \in \mathbb{R}} \|\phi_\epsilon(s, y(s))\| \leq \min \left\{ \frac{\beta\delta}{4M_1}, \frac{\beta^{1-\alpha}\delta}{4M_1\Gamma(1-\alpha)} \right\}. \quad (3.11)$$

By (3.10) and (3.11) we find that, for all $t \in \mathbb{R}$,

$$\begin{aligned} \|\mathcal{F}(y)(t)\|_\alpha &\leq \frac{2M_1}{\beta} \sup_{s \in \mathbb{R}} \|\phi_\epsilon(s, y(s))\| + M_1 \sup_{s \in \mathbb{R}} \|\phi_\epsilon(s, y(s))\| \int_{-\infty}^t e^{-\beta(t-s)} (t-s)^{-\alpha} ds \\ &\leq \frac{1}{2}\delta + \frac{\beta^{1-\alpha}}{4\Gamma(1-\alpha)}\delta \int_0^\infty e^{-\beta s} s^{-\alpha} ds \leq \delta, \end{aligned} \quad (3.12)$$

where we have used the integral $\int_0^\infty e^{-\beta s} s^{-\alpha} ds = \beta^{\alpha-1}\Gamma(1-\alpha)$ for $0 \leq \alpha < 1$. Note that (3.12) implies $\mathcal{F}(y) \in C_b(\mathbb{R}, X^\alpha)$ with norm $\|\mathcal{F}(y)\|_{C_b(\mathbb{R}, X^\alpha)} \leq \delta$. We now prove $\mathcal{F}(y)$ is almost periodic. If $y \in Z$, then $y : \mathbb{R} \rightarrow X^\alpha$ is almost periodic and hence the set $\{y(t) : t \in \mathbb{R}\}$ is precompact in X^α . As a consequence of this, it follows from [14] (Theorem 2.7, page 16) that the functions $g_\epsilon(\cdot, x_0^* + y(\cdot))$ and $h(y(\cdot))$ are almost periodic functions with values in X . Therefore, $\phi_\epsilon(t, y(t)) = h(y(t)) + g_\epsilon(t, y(t))$ is also almost periodic in X . By definition, given $\eta > 0$, there is a positive number l (depending on η) such that every interval I of length l contains a number σ for which

$$\|\phi_\epsilon(t + \sigma, y(t + \sigma)) - \phi_\epsilon(t, y(t))\| < \eta, \quad \forall t \in \mathbb{R}. \quad (3.13)$$

Using (2.5)-(2.6) and (3.13), from (3.6) we obtain, for all $t \in \mathbb{R}$,

$$\begin{aligned}
& \|\mathcal{F}(y)(t+\sigma) - \mathcal{F}(y)(t)\|_\alpha \\
& \leq \left\| \int_{-\infty}^{t+\sigma} e^{-A(t+\sigma-s)}(I-P)\phi_\epsilon(s, y(s))ds - \int_{-\infty}^t e^{-A(t-s)}(I-P)\phi_\epsilon(s, y(s))ds \right\|_\alpha \\
& \quad + \left\| \int_{t+\sigma}^\infty e^{-A(t+\sigma-s)}P\phi_\epsilon(s, y(s))ds - \int_t^\infty e^{-A(t-s)}P\phi_\epsilon(s, y(s))ds \right\|_\alpha \\
& \leq \left\| \int_{-\infty}^t e^{-A(t-s)}(I-P)(\phi_\epsilon(s+\sigma, y(s+\sigma)) - \phi_\epsilon(s, y(s)))ds \right\|_\alpha \\
& \quad + \left\| \int_t^\infty e^{-A(t-s)}P(\phi_\epsilon(s+\sigma, y(s+\sigma)) - \phi_\epsilon(s, y(s)))ds \right\|_\alpha \\
& \leq M_1 \int_{-\infty}^t e^{-\beta(t-s)}(1+(t-s)^{-\alpha})\|\phi_\epsilon(s+\sigma, y(s+\sigma)) - \phi_\epsilon(s, y(s))\|ds \\
& \quad + M_1 \int_t^\infty e^{\beta(t-s)}\|\phi_\epsilon(s+\sigma, y(s+\sigma)) - \phi_\epsilon(s, y(s))\|ds \\
& \leq \eta M_1 \int_{-\infty}^t e^{-\beta(t-s)}(1+(t-s)^{-\alpha})ds + \eta M_1 \int_t^\infty e^{\beta(t-s)}ds \\
& \leq \eta M_1 (2\beta^{-1} + \beta^{\alpha-1}\Gamma(1-\alpha)),
\end{aligned}$$

which shows that $\mathcal{F}(y) : \mathbb{R} \rightarrow X^\alpha$ is almost periodic. By (3.12) we see that \mathcal{F} given by (3.6) maps Z into itself. We next show that $\mathcal{F} : Z \rightarrow Z$ is a contraction.

Let $y_1, y_2 \in Z$. By (3.6) and (2.5)-(2.6) we have

$$\begin{aligned}
& \|\mathcal{F}(y_1)(t) - \mathcal{F}(y_2)(t)\|_\alpha \\
& \leq \left\| \int_{-\infty}^t e^{-A(t-s)}(I-P)(\phi_\epsilon(s, y_1(s)) - \phi_\epsilon(s, y_2(s)))ds \right\|_\alpha \\
& \quad + \left\| \int_t^\infty e^{-A(t-s)}P(\phi_\epsilon(s, y_1(s)) - \phi_\epsilon(s, y_2(s)))ds \right\|_\alpha \\
& \leq M_1 \int_{-\infty}^t e^{-\beta(t-s)}(1+(t-s)^{-\alpha})\|\phi_\epsilon(s, y_1(s)) - \phi_\epsilon(s, y_2(s))\|ds \\
& \quad + M_1 \int_t^\infty e^{\beta(t-s)}\|\phi_\epsilon(s, y_1(s)) - \phi_\epsilon(s, y_2(s))\|ds.
\end{aligned} \tag{3.14}$$

It follows from (3.7) and (3.9) that

$$\|\phi_\epsilon(s, y_1(s)) - \phi_\epsilon(s, y_2(s))\|$$

$$\begin{aligned}
&\leq \|h(y_1(s)) - h(y_2(s))\| + \|g_\epsilon(s, x_0^* + y_1(s)) - g_\epsilon(s, x_0^* + y_2(s))\| \\
&\leq \sup_{\|x\|_\alpha \leq \delta} \|h'(x)\|_{L(X^\alpha, X)} \|y_1(s) - y_2(s)\|_\alpha \\
&\quad + \sup_{s \in \mathbb{R}} \sup_{\|x\|_\alpha \leq \|x_0^*\|_\alpha + 1} \left\| \frac{\partial g_\epsilon}{\partial x}(s, x) \right\|_{L(X^\alpha, X)} \|y_1(s) - y_2(s)\|_\alpha \\
&\leq \frac{1}{M_1(4\beta^{-1} + 2\beta^{\alpha-1}\Gamma(1-\alpha))} \|y_1(s) - y_2(s)\|_\alpha.
\end{aligned} \tag{3.15}$$

By (3.14) and (3.15) we obtain, for all $t \in \mathbb{R}$,

$$\begin{aligned}
&\|\mathcal{F}(y_1)(t) - \mathcal{F}(y_2)(t)\|_\alpha \\
&\leq \frac{1}{(4\beta^{-1} + 2\beta^{\alpha-1}\Gamma(1-\alpha))} \sup_{s \in \mathbb{R}} \|y_1(s) - y_2(s)\|_\alpha \left(2 \int_0^\infty e^{-\beta s} ds + \int_0^\infty e^{-\beta s} s^{-\alpha} ds \right) \\
&\leq \frac{1}{(4\beta^{-1} + 2\beta^{\alpha-1}\Gamma(1-\alpha))} (2\beta^{-1} + \beta^{\alpha-1}\Gamma(1-\alpha)) \sup_{s \in \mathbb{R}} \|y_1(s) - y_2(s)\|_\alpha,
\end{aligned}$$

which shows that

$$\sup_{t \in \mathbb{R}} \|(\mathcal{F}(y_1) - \mathcal{F}(y_2))(t)\|_\alpha \leq \frac{1}{2} \sup_{t \in \mathbb{R}} \|(y_1 - y_2)(t)\|_\alpha,$$

and hence $\mathcal{F} : Z \rightarrow Z$ is a contraction. By the fixed point theorem, \mathcal{F} has a unique fixed point y_ϵ^* in Z , which implies that $x_\epsilon^* = x_0^* + y_\epsilon^*$ is the unique almost periodic solution of equation (2.8) satisfying $\sup_{t \in \mathbb{R}} \|x_\epsilon^*(t) - x_0^*\|_\alpha \leq \delta$. Taking $\delta = \delta_0$ we get the first part of Lemma 3.1. For arbitrary $\delta \in (0, \delta_0]$, the above process shows that

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in \mathbb{R}} \|x_\epsilon^*(t) - x_0^*\|_\alpha = 0,$$

which completes the proof. \square

We are now in a position to prove our main results.

Proof of Theorem 2.1. Given $\delta > 0$ and $x_0 \in X^\alpha$, denote by $\bar{B}_\alpha(x_0, \delta)$ the closed ball in X^α with center x_0 and radius δ , that is,

$$\bar{B}_\alpha(x_0, \delta) = \{x \in X^\alpha : \|x - x_0\|_\alpha \leq \delta\}.$$

Since equation (2.1) has only a set of finitely many equilibrium solutions x_i^* ($1 \leq i \leq n$), there is $\delta_0 > 0$ such that $\bar{B}_\alpha(x_i^*, \delta_0) \cap \bar{B}_\alpha(x_j^*, \delta_0) = \emptyset$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. Furthermore, for each $i \in \{1, \dots, n\}$, by Lemma 3.1 there exist $\delta_i \in (0, \delta_0)$ and $\epsilon_i > 0$ such that for every $\epsilon \in (0, \epsilon_i)$, equation (2.8) has a unique almost periodic solution $\phi_{i,\epsilon}^*$ for which $\phi_{i,\epsilon}^*(t) \in \bar{B}_\alpha(x_i^*, \delta_i)$ for all $t \in \mathbb{R}$.

Let $\epsilon_0 = \min\{\epsilon_i : 1 \leq i \leq n\}$. Then, for every $\epsilon \in (0, \epsilon_0)$, we want to show that $\phi_{i,\epsilon}^*$, $1 \leq i \leq n$, are the only almost periodic solutions of equation (2.8). Suppose $x_\epsilon^* : \mathbb{R} \rightarrow X^\alpha$ is an arbitrary almost periodic solution of equation (2.8). Since an almost periodic solution is a complete bounded solution, it follows from [2] that there is $i \in \{1, \dots, n\}$ such that

$$\lim_{t \rightarrow \infty} \|x_\epsilon^*(t) - \phi_{i,\epsilon}^*(t)\|_\alpha = 0.$$

Therefore, given $\eta > 0$, there is $T > 0$ such that for all $t \geq T$, the following holds:

$$\|x_\epsilon^*(t) - \phi_{i,\epsilon}^*(t)\|_\alpha \leq \eta. \quad (3.16)$$

On the other hand, since x_ϵ^* is almost periodic, for the given $\eta > 0$, there is a positive number l (depending on η) such that every interval I of length l contains a number s for which

$$\|x_\epsilon^*(\tau + s) - x_\epsilon^*(\tau)\|_\alpha \leq \eta, \quad \forall \tau \in \mathbb{R}.$$

This implies that for every $t \in \mathbb{R}$, there is a number $s_0 \in [T - t, T - t + l]$ such that

$$\|x_\epsilon^*(t + s_0) - x_\epsilon^*(t)\|_\alpha \leq \eta. \quad (3.17)$$

By $s_0 \in [T - t, T - t + l]$ we have $t + s_0 \in [T, T + l]$ and hence it follows from (3.16) that

$$\|x_\epsilon^*(t + s_0) - \phi_{i,\epsilon}^*(t + s_0)\|_\alpha \leq \eta. \quad (3.18)$$

By (3.17) and (3.18) we obtain

$$\|x_\epsilon^*(t) - \phi_{i,\epsilon}^*(t + s_0)\|_\alpha \leq 2\eta. \quad (3.19)$$

Since $\phi_{i,\epsilon}^*(t) \in \bar{B}_\alpha(x_i^*, \delta_i)$ for all $t \in \mathbb{R}$, we find from (3.19) that, for all $t \in \mathbb{R}$,

$$\|x_\epsilon^*(t) - x_i^*\|_\alpha \leq \delta_i + 2\eta, \quad \forall \eta > 0.$$

Taking $\eta \rightarrow 0$, we see that x_ϵ^* is an almost periodic solution of equation (2.8) which satisfies $x_\epsilon^*(t) \in \bar{B}_\alpha(x_i^*, \delta_i)$ for all $t \in \mathbb{R}$. Note that $\phi_{i,\epsilon}^*$ is the unique almost periodic solution of the equation which belongs to $\bar{B}_\alpha(x_i^*, \delta_i)$. Therefore we must have $x_\epsilon^*(t) = \phi_{i,\epsilon}^*(t)$ for all $t \in \mathbb{R}$. Since x_ϵ^* is an arbitrary almost periodic solution, we see that the non-autonomous equation has no any other almost periodic solutions except $\phi_{i,\epsilon}^*$, $1 \leq i \leq n$. This concludes the proof of Theorem 2.1.

As mentioned earlier, Corollary 2.2 was proved by Carvalho et. al. in [2] when $\phi_{i,\epsilon}^*$, $1 \leq i \leq n$, are complete bounded solutions. In the present paper, we demonstrate that these solutions are actually almost periodic solutions under almost periodic solutions. Therefore, Corollary 2.2 is an immediate consequence of Theorem 2.1 and the results of [2].

4 Applications

In this section, we discuss an application of our results to the Chafee-Infante equation and describe the almost periodic dynamics of the non-autonomous equation under small perturbations. The one-dimensional Chafee-Infante equation reads

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \lambda(u - u^3), \quad x \in (0, \pi), \quad t > 0, \quad (4.1)$$

with the boundary condition

$$u(t, 0) = u(t, \pi) = 0, \quad t > 0, \quad (4.2)$$

and the initial condition

$$u(0, x) = u_0(x), \quad x \in (0, \pi), \quad (4.3)$$

where λ is a positive parameter.

Let $A_0 = -\partial_{xx}$ with domain $D(A_0) = H^2((0, \pi)) \cap H_0^1((0, \pi))$. Then A_0 is a sectorial operator in $X = L^2((0, \pi))$ and the eigenvalues of A_0 are given by $\lambda_n = n^2$ where n is any positive integer. It is well known that problem (4.1)-(4.3) is well-posed in $D(A_0^{\frac{1}{2}}) = H_0^1((0, \pi))$. In other words, system (4.1)-(4.3) defines a continuous semigroup $\{S(t)\}_{t \in \mathbb{R}}$ on $H_0^1((0, \pi))$ (see, e.g., [16]). This semigroup has a global attractor \mathcal{A} in $H_0^1((0, \pi))$. The structure of equilibrium solutions of problem (4.1)-(4.3) is well understood. Actually, for every $\lambda \in (n^2, (n+1)^2)$ where n is any nonnegative integer, it was proved by Chafee and Infante in [17] that problem (4.1)-(4.3) has exactly $2n+1$ equilibrium solutions. It was further proved by Henry in [18] that all these equilibrium solutions are hyperbolic. Let $V : H_0^1((0, \pi)) \rightarrow \mathbb{R}$ be given by

$$V(u) = \int_0^\pi \left(\left(\frac{\partial u}{\partial x} \right)^2 - \lambda u^2 + \frac{1}{2} \lambda u^4 \right) dx, \quad \forall u \in H_0^1((0, \pi)).$$

Then V is a Liapunov function of $\{S(t)\}_{t \in \mathbb{R}}$. So assumptions (H1)-(H3) are all fulfilled in this case, and the attractor \mathcal{A} is given by the union of unstable manifolds of the $(2n+1)$ equilibrium solutions. Further the Hausdorff dimension of \mathcal{A} is n as shown in [16].

Suppose $g : \mathbb{R} \rightarrow \mathbb{R}$ is an almost periodic function and $h \in L^2((0, \pi))$. Given $\epsilon > 0$, consider now the non-autonomously perturbed equation, for all $\tau \in \mathbb{R}$,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \lambda(u - u^3) + \epsilon g(t)h(x), \quad x \in (0, \pi), \quad t > \tau, \quad (4.4)$$

with the boundary condition

$$u(t, 0) = u(t, \pi) = 0, \quad t > \tau, \quad (4.5)$$

and the initial condition

$$u(\tau, x) = u_0(x), \quad x \in (0, \pi), \quad (4.6)$$

Set $g_\epsilon(t, x) = \epsilon g(t)h(x)$ for all $t \in \mathbb{R}$ and $x \in (0, \pi)$. Then it is evident that g_ϵ satisfies condition (H4). Given $\epsilon > 0$, the existence of pullback attractor $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$ for problem (4.4)-(4.6) in $H_0^1((0, \pi))$ can be proved by standard arguments, see, e.g., [5, 19]. Note that $g : \mathbb{R} \rightarrow \mathbb{R}$ is bounded since it is almost periodic. Then it is easy to verify that the attractor $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$ indeed satisfies condition (H5) with $\epsilon_0 = 1$. Applying Theorem 2.1 and Corollary 2.2 to system (4.4)-(4.6) we have

Theorem 4.1. *Let n be a nonnegative integer and $\lambda \in (n^2, (n+1)^2)$. Then There is $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, problem (4.4)-(4.6) has exactly $2n+1$ almost periodic solutions $\phi_{i,\epsilon}^* : \mathbb{R} \rightarrow H_0^1((0, \pi))$, $1 \leq i \leq 2n+1$. The system has an n -dimensional pullback attractor $\{\mathcal{A}_\epsilon(t)\}_{t \in \mathbb{R}}$ which is the union of the unstable manifolds of the almost periodic solutions. Further, every solution of problem (4.4)-(4.6) converges to one of those almost periodic solutions.*

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